ON THE USE OF GENERALISED BEAM THEORY TO ASSESS THE BUCKLING AND POST-BUCKLING BEHAVIOUR OF LAMINATED CFRP CYLINDRICAL STIFFENED PANELS

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Abstract
The paper presents the application of a novel fast numerical tool, based on Generalised Beam Theory (GBT), to perform buckling and post-buckling analyses of laminated CFRP panels. GBT is a beam theory developed for prismatic thin-walled members (e.g., columns, beams or panels), which takes into account both global and local deformations. One of its main features is the fact that the cross-section is discretised into deformation modes with clear mechanical meanings (e.g., global bending, distortional, local-plate, shear and transverse extension deformation modes) – this (i) allows for a better understanding of the member structural behaviour and (ii) makes it possible to perform analyses with very few d.o.f. (by pre-selecting a set of deformation modes). No stiffness degradation is taken into consideration and the material is deemed linear elastic and orthotropic. One presents numerical results concerning the local buckling and post-buckling behaviour of stiffened CFRP cylindrical panels, including one that was experimental and numerically investigated in the context of the COCOMAT project. The panel deformed configurations and buckling loads obtained with GBT are validated through the comparison with either experimental data or values yielded by shell finite element analyses carried out in the code ABAQUS.

1 Introduction
An increasing interest of the aerospace industry for composite material solutions led to great improvements in the laminated FRP fabrication processes and to the development of more sophisticated numerical tools to render their design as efficient as possible. In particular, there is a high demand for design tools that are capable of performing fast and reliable buckling and post-buckling analyses of laminated FRP cylindrical panels. Generalised Beam Theory (GBT) can be viewed as either (i) a beam theory that incorporates local deformations or (ii) a folded-plate theory which takes into account global deformation. Its main feature resides in the fact that the member cross-section is discretised into global (axial extension, bending and torsion), local (local-plate and distortional), shear and transverse extension deformation modes as opposed to the nodal displacements and rotations adopted by the traditional shell finite element (FE) models. Although the first publication on GBT appeared more than four decades ago (Schardt), it was only very recently that the first GBT formulations for laminated composite materials were developed (e.g., Silvestre and Camotim and Silva et al.). The aim of this work is to illustrate the application of GBT linear (first-order), buckling and post-buckling formulations to assess the structural behaviour of fixed laminated CFRP stiffened cylindrical panels subjected to axial compression. The GBT buckling loads, buckling modes and deformed configurations are validated through the comparison with either (i) values yielded by shell FE analyses, carried out in the code ABAQUS, or (ii) experimental data obtained in the context of the COCOMAT Project.

2 GBT Fundamentals
Consider a laminated fibre reinforced plastic (FRP) prismatic beam or panel with an arbitrary thin-walled cross-section – the plate local coordinates are (i) longitudinal/axial (x), (ii) transverse, along the cross-section mid-line (s), and (iii) normal to the plate thickness (z). In GBT, the corresponding displacement field components u, v and w (plates mid-line) read

\[ u(x,s) = u_k(s) \phi_k(x), \quad v(x,s) = v_k(s) \phi_k(x), \quad w(x,s) = w_k(s) \phi_k(x), \]  

(2.1)

where (i) subscript \( \cdot \) denotes differentiation w.r.t. x, (ii) \( u_k(s) \), \( v_k(s) \) and \( w_k(s) \) are shape functions defining deformation mode \( k \) and (iii) \( \phi_k(x) \) is the corresponding modal amplitude function, providing the variation along the member axis. A GBT analysis
comprises (i) a cross-section analysis (see Silva et al.⁶), in order to determine the deformation modes and evaluate the associated mechanical properties, and (ii) a member (or panel) analysis, where the global linear, buckling or post-buckling problem is solved (see Silva et al.³).

2.1 GBT finite element
The panel analyses are performed by means of a special beam finite element in which the degrees of freedom (d.o.f.) are the nodal values of the GBT deformation mode amplitudes. The shape functions $\psi_\alpha(x)$ adopted to discretise the longitudinal modal amplitudes functions $\phi(x)$ are either (i) Hermite (bending, torsion, local and transverse extension modes) or (ii) Lagrange (axial extension and warping shear modes) cubic polynomials (see Silvestre and Camotim²).

2.2 Panel non-linear analysis: incremental-iterative approach
In order to determine the panel post-buckling equilibrium paths, one must perform geometrically non-linear analyses, i.e., solve systems of non-linear equations. This task is performed in this work by means of an incremental-iterative technique based on the well known Newton-Raphson’s method and employing an arc-length control strategy – the tangent stiffness matrix and internal force vector are calculated (updated) after each iteration. Due to space limitations, the details of this incremental-iterative approach are not addressed here – the details can be found in Silva et al.³ and the adopted methodology follows closely the one developed by Silvestre and Camotim² in the context of cold-formed steel members. As for the GBT buckling analyses, they are carried out employing a geometric stiffness matrix derived on the basis of first-order results and post-buckling tensor quantities.

3 Illustrative Examples

3.1 Panel 1 – Design 1 panels from the COCOMAT project
The first illustrative example deals with the buckling behaviour of a fixed CFRP stiffened cylindrical panel subjected to an axial compressive load ($P$). The panel, which is identical to the Design 1 panels P23, P28 and P29 investigated in the context of the COCOMAT project⁴, has a free buckling length $L=660$ mm, radius $r=1000$ mm, arc-length $a=560$ mm and ply thickness $t_{ply}=0.125$ mm. It contains five T-shaped stringers attached to the inner face and have $b_s=32$ mm widths and $h_s=14$ mm heights. The stacking sequence is (i) $[90/+45/-45/0]_S$ in the skin, (ii) $[(+45/-45)3/06]_S$ in the blades (stringer web) and (iii) $[(+45/-45)_3/0]_S$ in the stringer flanges – they are made of a prepreg IM7/8552 material with elastic properties $E_1=164.1$ GPa, $E_2=9.2$ GPa, $v_{12}=0.28$ and $G_{12}=5.1$ GPa. The panel buckling behaviour was determined by means of GBT analyses and its discretisation involved 32 nodes in the cross-section and 12 beam finite elements along the length. After performing the GBT cross-section analysis, one was led to 96 deformation modes, the most relevant of which are displayed in Fig. 1(a) – global (1-4), distortional (5-9), local (10-32), shear (33-65) and transverse extension (66-96) modes. The GBT-based critical buckling mode shape is shown in Fig. 1(b) and can be compared with the experimental one, presented in Fig. 1(c) and obtained by means of the ARAMIS system (photogrammetry) at a shortening of $\delta=0.72$ cm. On the other hand, Figs. 1(d) and 1(e) show (i) the amplitude functions $\phi(x)$ of the most important deformation modes, concerning the panel critical
buckling mode shape, and (ii) the experimental load-shortening curves of the COCOMAT panel P28. The observation of these buckling results prompts the following comments:

(i) The critical buckling mode is local and exhibits 10 longitudinal and 4 transverse half-waves – there is a remarkable similarity between the GBT-based and experimental mode shapes. The critical buckling load obtained \( P_{cr,exp} = 52.1 \text{kN} \) is also very close to the experimental one \( P_{cr,exp} = 50 \text{kN} \) – difference below 5% . It is worth noting that \( P_{cr,exp} \) is the load associated with a visible slope change (drop) occurring in the panel load-axial shortening curve.

(ii) The panel critical buckling mode shape combines contributions from the (i) distortional modes \( 9 (25.3\%), 5 (8.0\%) \) and \( 7 (3.2\%) \), (ii) local modes \( 12 (7.9\%), 17 (7.1\%) \) and \( 19 (5.2\%) \), (iii) shear mode \( 40 (3.4\%) \) and (iv) transverse extension mode \( 66 (7.9\%) \) – this unique GBT modal decomposition provides clear insight into the panel buckling mechanics. Since the GBT code “used in this work is not yet numerically optimised, no post-buckling results are presented for this panel – it would take too long to obtain them”.

3.2 Panel 2 – simplified panel

The second illustrative example concerns a CFRP cylindrical panel with the geometry, loading, boundary conditions and material properties of the previous one but having (i) only two stiffeners (arc-length \( a=164 \text{ mm} \) (ii) length \( L=330 \text{ mm} \) and (iii) a stacking sequence \([90/+45/-45/0]_S \) in all walls. A cross-section discretisation involving 11 nodes was adopted, leading to 33 GBT deformation modes – Fig. 3 (d) shows the most relevant ones: global (1-4), local (5-13), shear (14-23) and transverse extension (24-33).

Figs. 2 (a)-(b) show panel deformed configurations obtained with first-order (geometrically linear) GBT and ABAQUS (shell FE) analyses corresponding to a \( P=1 \text{kN} \) compressive load – note that both the axial shortenings and deformed configurations are remarkably similar. Figs. 2 (c)-(d), on the other hand, display the GBT and ABAQUS panel critical buckling mode shapes – they are also extremely similar and the corresponding critical buckling loads only differ by about 1.0%.

![Fig. 2 Panel 2 (a+b) first-order deformed configurations at \( P=1 \text{kN} \) (amplified 2000 times) and (c+d) critical buckling mode shapes (provided by GBT and ABAQUS analyses, respectively)](image)

The panel was also analysed by means of the GBT post-buckling formulation and methodology mentioned in section 2.2. The initial arc-length was set to \( l_0=0.01 \text{ cm} \) and is continuously adapted along the incremental loading process – the analysis was carried out until a shortening \( \delta=0.36 \text{ mm} \) was reached 13. Three different GBT analyses were performed, differing in the number of deformation modes included: (i) all 33 deformation modes, (ii) only the 18 symmetric ones \((1,3,5,7,9,11,13,15,17,19,21,23,24,26,28,30,31,32)\) and (iii) only a selection of 13 of them \((1,3,5,7,9,11,13,15,19,21,23,24,26)\) – the corresponding panel d.o.f. numbers are 870, 475 and 352. While Fig. 3(a) shows the load-shortening curves for the yielded by the three analyses, Fig. 3(b) depicts the GBT-based panel deformed configuration at \( \delta=0.36 \text{mm} \) (18 modes) and Fig. 3(c) displays a modal participation diagram that provides the evolution, along the post-buckling path, of the percentage contribution of each deformation mode to the panel deformed deformation along the post-buckling path. After observing the presented results, the following remarks seem appropriate:

1. Although the total number of d.o.f. required to reach convergence in GBT buckling analyses is much lower than in ABAQUS shell FE ones (1200 vs 6000), the former use text files to store the 4th order tensors (output of the GBT cross-section analysis) – they currently occupy more than 25 times the “optimal memory” required, which obviously leads to a dramatic loss in computational efficiency (the GBT analyses become as slow as the ABAQUS ones, or even slower). This problem does not occur in members with simple cross-sections defined by straight walls (e.g., lipped channel or I-section members), which involve much less d.o.f. The optimisation of the GBT tools for complex panels will be done in the near future. Note, however, that this limitation can be partially overcome by including only few deformation modes in the (approximate) GBT analyses (see section 3.2) – this possibility is a unique GBT feature.

2. The unusual deformed configuration with significant transversal bending near the supports is due to a combination of (i) Poisson’s effect, (ii) the panel curvature and (iii) the fixed boundary conditions – this same phenomenon would occur even in the case of an isotropic panel.

3. It was not possible to carry the analysis beyond this point due to numerically instabilities, apparently caused by the arc-length control strategy – this same problem is encountered in shell finite element analyses that employ arc-length control. Overcoming these difficulties may require the inclusion of an “artificial damping” – this issue will be addressed in the near future.

4. No initial geometrical imperfection is included in the analyses, due to the fact that the pre-buckling deformed configuration already involves local deformations – although the inclusion of a critical-mode initial geometrical imperfection would lead to more conservative results, this more simple approach was adopted.
(i) The panel post-buckling strength reserve is very significant – it amounts to at least 80% of the critical buckling load (recall that, due to insurmountable numerical instabilities, the post-buckling analysis could not be carried out further).

(ii) The panel deformed configuration at $\delta=0.36\,\text{mm}$, displayed in Fig. 3(b), combines contributions from deformation modes 1 (13.5%), 3 (8.9%), 5 (49.9%), 7 (4.4%), 9 (8.6%), 11 (2.9%), 15 (2.4%), 19 (2.0%) and 24 (4.6%). The modal participation diagram presented in Fig. 3(c) shows how these contributions vary along the equilibrium path. One observes that the participation of mode 1 (axial extension) decreases, while the opposite occurs for the contributions of modes 3 (minor axis bending) and 5 (local). The abrupt slope change taking place at $\delta=0.17\,\text{mm}$ indicates the occurrence of local buckling.

(iii) The significant d.o.f. reduction from 870 (33 modes) to 352 (13 modes) practically does not affect the results obtained. The possibility of reducing the d.o.f. number, while controlling the accuracy of the analysis, is a GBT trademark – since some deformation modes play less roles than others, their exclusion from the analysis has minute consequences.

![Graph showing load-shortening curves](image)

![Diagram showing modal participation](image)

**Fig. 3** Panel 2 (a) load-shortening curves, (b) deformed configuration at $\delta=0.36\,\text{mm}$ (amplified 20 times), (c) modal participation diagram and (d) most relevant GBT deformation mode shapes

4 Conclusion
This paper addressed the application of a recently developed GBT-based numerical tool to analyse the first-order, buckling and post-buckling behaviours of stiffened FRP cylindrical panels. The main conclusions of this study are:

(i) The GBT analyses provided panel first-order and buckling results that are in very good agreement with the values observed in experimental tests and/or yielded by ABAQUS shell finite element analyses.

(ii) The GBT unique modal output provided a clear insight into the structural responses of the analysed panels – e.g., it was possible to quantify the individual contributions of the global, local, shear and transverse extension deformation modes.

(iii) The possibility to drastically reduce the number of d.o.f. with a minute accuracy loss is another advantage of the GBT analyses, proves its highly promising potential as a fast tool for the design of composite panels.

References
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